Steady Flow through the Unsaturated Zone

Darcy's Law: $$q = -K\frac{dH}{dz}$$

where $H = h + z$ (z axis points upwards)

and $$h = \frac{p - p_{atmos}}{pg}$$ is the pressure head

$$\Rightarrow \quad q = -K\frac{d}{dz}(h + z) = -K\left(\frac{dh}{dz} + 1\right)$$

Note:

$$\left.\begin{array}{l} q > 0 \iff upward\ flow \\ q < 0 \iff downward\ flow \end{array}\right\} \quad z \text{ axis points upwards}$$

Special cases

I. No flow $\iff q = 0 \iff$ hydrostatic

or

$$\frac{dh}{dz} = -1 \implies h \text{ has the form:}$$

$h = -z + A$ (i.e., linear in z), where A is a constant

II. Uniform moisture content in the soil profile $\Rightarrow h$ is constant with z

$$\implies \frac{dh}{dz} = 0$$

$$\implies q = -K \text{ (flow by gravity only)}$$

Note that here, $q = -K\ (h_c)$ where h_c is constant and corresponds to the moisture content.

General solution for steady unsaturated flow

$$q = -K(h)\left(\frac{dh}{dz} + 1\right)$$

so $$\frac{dh}{dz} = -\frac{q}{K(h)} - 1$$

$$\therefore \qquad \frac{dh}{\frac{q}{K(h)} + 1} = -dz$$

$$\therefore \qquad \int_0^h \frac{d\bar{h}}{\frac{q}{K(\bar{h})} + 1} = -\int_0^z d\bar{z} = -z$$

where we assume $h(0) = 0$ (e.g., the origin of the z axis is located at the watertable).

If $h(0) = h_0 \neq 0$, then the expression becomes:

$$\int_{h_0}^h \frac{d\bar{h}}{\frac{q}{K(\bar{h})} + 1} = -z$$

If $K(h) = K_s \exp(ah)$ $(h \leq 0, a > 0)$ then the integral becomes:

$$\frac{1}{a} \ln[K_s \exp(ah) + q]_0^h = -z$$

$$\therefore \qquad z = \frac{1}{a} \ln\left[\frac{K_s + q}{K_s \exp(ah) + q}\right]$$

Solve for h : $$h(z) = \frac{1}{a} \ln\left[\frac{(K_s + q)\ \exp(-az) - q}{K_s}\right]$$

When $q > 0$ (evaporation), then the ln term will reach ≤ 0. Since $\ln(0) = -\infty$, this indicates the soil is completely dry. This will occur when:

$$(K_s + q) \exp(-az) - q = 0$$

or

$$z = \frac{1}{a}\ln\left(\frac{K_s + q}{q}\right)$$

For a given evaporation flux, this gives the maximum height to which water will rise above the watertable. Note that the water flux is unchanged, which means that the water is evaporating (although the change of phase is not considered by the model).

Steady Flow through the Saturated Zone

For $h > 0, K = K_s$, not $K_s \exp(ah)$, so the solution must be modified for this case. The solution is:

$$\int_0^h \frac{d\bar{h}}{\frac{q}{K_s} + 1} = -z \qquad h > 0$$

$$\therefore \qquad h = -z\left(\frac{q}{K_s} + 1\right)$$

Since $h > 0$, this solution is valid for

$$-z\left(\frac{q}{K_s} + 1\right) > 0$$

or

$$\frac{q}{K_s} + 1 < 0$$

or

$$q < -K_s$$

The physical interpretation of this case is that the soil profile is saturated, with a certain height of water on the soil surface, which results in a flux q with a magnitude greater than K_s. If the depth of waer on the surface is 0 then $q = -K_s$. and $h = 0$ everywhere. Note that the boundary condition at the watertable is still $h = 0$. This is probably physically unreasonable, although physically it could occur if the water reaching the watertable is rapidly move horizontally as aquifer flow. It is more likely that the water reaching the watertable would be moved more slowly, which would result in water collecting at/above the watertable – this is called mounding. In this case, the boundary condition at the watertable would not satisfy $h = 0$, and so the above solution would not be valid.